Superintegrable models of Winternitz type.

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Abstract

A general procedure is outlined which allows to construct superintegrable models of Winternitz type. Some examples are presented.

Due to their exceptional properties the superintegrable systems, both classical and quantum, are subject of constant interest. The number of examples is here, however, slightly limited. The aim of the present paper is to outline some general procedure leading to superintegrable models generalizing Winternitz system [1] $\div$ [4].
Let us consider an integrable classical system of N degrees of freedom coupled by confining forces. According to the general theory [5] one can then define action-angle variables $J_k, \varphi_k, \quad k = 1, ..., N$; the submanifolds $J_k = const$ are (sum of) invariant Liouville-Arnold tori T^N, parametrized by the angels φ_k provided the identification $T^N \sim (S^1)^N$ has been made.
Generically, when restricted to invariant torus, the dynamics appears to be ergodic. If this is not the case the system posses an additional integral of motion which is independent of J_k's. Such a system is called superintegrable. The maximal number of these additional integrals is $N - 1$ and the corresponding dynamics is called maximally superintegrable.
It is easy to see that all trajectories of maximally superintegrable system, being compact common intersections of $2N - 1$ hypersurfaces, are closed. This

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is possible if and only if all ratios $\omega_k(J)/\omega_j(J)$ of the frequencies $\omega_k(J) \equiv \frac{\partial H}{\partial J_k}$ are rational numbers.This implies

$$\omega_k(J) = m_k \omega(J) \tag{1}$$

where m_k are integers and $\omega(J)$ is a fixed function of action variables.
The general form of the hamiltonian leading to (1) is

$$H(J) = H(\sum_{k=1}^{N} m_k J_k) \tag{2}$$

The additionals integrals of motion can be now easily constructed [6]. To this end let m be the least common multiple of m_k, $k = 1, ..., N$ and let $l_k \equiv m/m_k$. Then $\sin(l_k\varphi_k - l_1\varphi_1)$, (or cosines),$k = 2, ..., N$, are well-defined isolating independent integrals of motion;indeed,their time-independence is a direct consequence of eq.(1) while their single-valudness follows from the invariance under substitutions

$$\varphi_i \longrightarrow \varphi_i + 2\Pi n_i, n_i \in Z$$

Moreover,they are functionally independent exept nowhere dense set of points where some of the action variables vanish.It is easy to see that ,together with the action variables,these additional integrals define trajectories so any other time-independent integral is expressible in terms of them.This can be checked explicitly in the case of most prominent examples of superintegrable systems like Kepler problem, where the additional integrals are provided by Runge-Lenz vector, or Winternitz system.
As in the case of Winternitz system let us now start with completely separated hamiltonian

$$H = \sum_{k=1}^{N} (\frac{P_k^2}{2\mu_k} + U_k(x_k)) \equiv \sum_{k=1}^{N} H_k \tag{3}$$

which is immediately known to be integrable. However, even being so simple, H is generically not superintegrable. So one can pose the question which potentials $U_k(x)$ lead to superintegrable dynamics, in particular-maximally superintegrable. Taking into account that H is completely separated and using eq.(2) we conclude that H should be of the form

$$H = \alpha \cdot \sum_{k=1}^{N} m_k J_k \tag{4}$$

with some real constant α. This implies that all periods

$$T_k \equiv 2\pi \frac{dJ_k}{dH_k} = \frac{2\pi}{\alpha m_k} \tag{5}$$

are constant, i.e. energy-independent. This result is rather obvious: consider $\omega_k(E_k)/\omega_1(E_1)$ as a function of E_k (or E_1); it is continous and attains only rational values so it must be a constant.

We conclude that H, given by eq.(3), is maximally superintegrable if and only if all $U_k(x)$ are such that the corresponding periods of one-dimensional motions are energy-independent and their ratios are rational numbers. The solution to this problem is, however, well known [6]. Assume that $U(x)$ is such that (i) 0 is the absolute minimum of $U(x)$, (ii) $U(x) = E$ has exactly two solutions $x_{1,2}(E)$ for any $E > 0$. Then, given the period $T(E)$ as a function of energy, one can find all $U(x)$, which produce $T(E)$, from the equation

$$x_2(E) - x_1(E) = \frac{1}{\pi\sqrt{2\mu}} \int_0^E \frac{T(\varepsilon)d\varepsilon}{\sqrt{E-\varepsilon}} \tag{6}$$

In our case $T_k(\varepsilon) = 2\pi/\alpha m_k$ so $U_k(x)$ is given by

$$x_2(E) - x_1(E) = \frac{2}{\alpha m_k\sqrt{2\mu_k}} \int_0^E \frac{d\varepsilon}{\sqrt{E-\varepsilon}} = \frac{4}{\alpha m_k\sqrt{2\mu_k}}\sqrt{E} \tag{7}$$

Eq.(7) gives the general solution to the problem which hamiltonians H, eq.(3), are maximally superintegrable.

To find some particular classes of potentials U_k let us note that, due to the fact that there are exactly two solutions to $U_k(x) = E$ one can write

$$x_2(E) = \varphi_k(x_1(E)) \tag{8}$$

with some, yet unspecified, function $\varphi_k(x)$.

Then $U_k(x)$ takes the form

$$U_k(x) = \beta_k^2(\varphi_k(x) - x)^2, \;\; \beta_k^2 \equiv \frac{\alpha^2 m_k^2 \mu_k^2}{8} \tag{9}$$

The condition $U_k(x) = U_k(\varphi_k(x))$ will be obeyed if $\varphi_k(\varphi_k(x)) = x$. The choice $\varphi_k(x) = -x$, resp. $\varphi_k(x) = \gamma_k/x$, correponds to harmonic oscillators, resp. Winternitz system.

It is not difficult to find other examples. Let $\eta_k > 0$ be arbitrary real positives (of dimension of lenght) and let $\varphi_k(x)$

$$\varphi_k(x) = \frac{\eta_k x}{\sqrt{x^2 - \eta_k^2}} \tag{10}$$

Then $\varphi_k \circ \varphi_k = id$ and $U_k(x)$ defined by (9) obeys $(i), (ii)$. Therefore, the hamiltonian

$$H = \sum_{k=1}^{N} (\frac{p_k^2}{2\mu_k} + \beta_k^2 x_k^2 (1 - \frac{1}{\sqrt{(\frac{x_k}{\eta_k}^2)^2 - 1}})^2) \tag{11}$$

with $\eta_k < x_k < \infty$, is maximally superintegrable. In principle, one can construct the additional integrals of motion according to the recipe formulated above. However, they are not expressible in terms of simple (elementary, elliptic...) functions. This model can be easily generalized. Take n_k to be positive integer and write

$$\varphi_k(x) \equiv \eta_k x (x^{2n_k} - \eta_k^{2n_k})^{-\frac{1}{2n_k}}, \quad \eta_k < x < \infty \tag{12}$$

Again one checks easily that $\varphi_k \circ \varphi_k = id$ and $U_k(x)$ obeys (i) and (ii).
Although the explicit expression for additional integrals of motion are not accessible it is not difficult to generalize the Evans result [4] concerning the dynamical Poisson algebra for Winternitz system. It appears that for the systems defined above it is again $sp(2N, R)$ [7].
We have assumed above that there are exactly two solutions to the equation $U_k(x) = E$ because for this case the problem of finding $U_k(x)$ in terms of $T_k(E)$ is tractable in a simple way.However,this assumption seems to be also important on its own because we need T_k to be energy- independent.For the energies E close to the local minimum the frequency squared of the motion equals second derivative of the potential at the minimum. On the other hand,if $U(x) = E$ has more solutions there must be also local maximum of $U(x)$ and for the energy equal to the value of $U(x)$ at such a maximum the period becames infinite.In other words,the period can not be energy-independent.
The algorithm outlined above allows to produce many superintegrable systems of arbitrary number of degrees of freedom. It allows also in principle to construct the additional integrals of motion; one has to find the angle variables and to construct the appropriate single-valued functions of them.

It is also worth to note that eq.(6) allows to construct the models which are superintegrable in some region of phase space only. To this end we select for any k the energy intervals $E_k, E_k + \Delta_k$ and assume that $T_k(E_k) \equiv T_k$ are constant over intervals and T_k/T_l are rational. Eq.(6) allow us then to find the relevant potentials. The resulting hamiltonian is superintegrable in the above specified region of phase space.

References

[1] P.Winternitz et al., Sov. J. Nucl. Phys. 4 (1966), 444

[2] J.Fris et al., Phys. Lett. 16 (1965), 354

[3] A.Makarov et al., Nuovo Cimento 52 (1967), 1061

[4] N.Evans, Phys. Lett. 147A (1990), 483

[5] V.Arnold, Mathematical Methods of Classical Mechanics, Springer 1978

[6] L.Landau, E.Lifshitz, Mechanics, Pergamon Press 1976

[7] C.Gonera, M.Majewski, Acta Phys.Polon. B32 (2001),1167